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Asymptotic behavior of solutions of difference equations of second order

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Abstract

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For second-order linear and nonlinear difference equations some qualitative properties of solutions, like boundedness, asymptotic forms, existence of approximate zeros or first-order polynomial solutions, and also lack of the higher-order polynomial solutions, are studied.

Keywords: Second-order difference equation; asymptotic forms; sufficient conditions; comparison method.

In this paper we would like to present some elements of qualitative theory of difference equations. Asymptotic behavior of solutions of second-order difference equations will be investigated.

Let $\mathbb{N}$ denote the set of positive integers, $\mathbb{R}$ the set of real numbers and $\mathbb{R}_+$ the set of nonnegative reals. For a function $a : \mathbb{N} \to \mathbb{R}$ we introduce the difference operator Δ by

$$\Delta a_n = a_{n+1} - a_n, \quad \Delta^2 a_n = \Delta(\Delta a_n),$$

where $a_n = a(n)$, $n \in \mathbb{N}$.

Moreover, let $\sum_{j=k}^{k-1} a_j = 0$ and $\prod_{j=k}^{k-1} a_j = 1$. The sequence $\{x_n\}_{n=1}^{\infty}$ is called *oscillatory* if for every $n \in \mathbb{N}$ there exists m, $m > n$, such that $x_m x_{m+1} \leqslant 0$. Otherwise the sequence is called *nonoscillatory*.

1. Bounds for second-order linear equations

To start with, the second-order linear difference equation

$$\Delta^2 y_n = p_n y_n, \quad n \in \mathbb{N}, \tag{E1}$$

will be considered. Some properties of solutions of this equation have been investigated in many papers, for instance, in [3].

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In the first theorem a bound of nonoscillatory solution of difference equation (E1) is given. It will not be assumed that the sequence of coefficients $\{p_n\}_{n=1}^{\infty}$ is of the same sign.

Theorem 1.1. *Let $\{p_n\}_{n=1}^{\infty}$ be such that the series*

$$\sum_{j=1}^{\infty} p_j \text{ is convergent}, \quad i.e., \sum_{j=1}^{\infty} p_j = g, \tag{1}$$

and let $g > 0$. Moreover, let n_1 exist such that

$$g - \sum_{j=1}^{n-1} p_j > 0, \quad \text{for all } n \geqslant n_1.$$

Then for every nonoscillatory solution y of (E1) *there exist a constant C and $n_0 \in \mathbb{N}$ such that*

$$\text{(i)} \quad \Delta y_n \leqslant C \prod_{j=n_0}^{n} (1 + P_j),$$

if y is eventually positive;

$$\text{(i}_1\text{)} \quad y_n \leqslant \frac{C}{P_n} \prod_{j=n_0}^{n} (1 + P_j),$$

if moreover y is nondecreasing;

$$\text{(ii)} \quad \Delta y_n \geqslant C \prod_{j=n_0}^{n} (1 + P_j),$$

if y is eventually negative;

$$\text{(ii}_1\text{)} \quad y_n \geqslant \frac{C}{P_n} \prod_{j=n_0}^{n} (1 + P_j),$$

if moreover y is nonincreasing, for $n \geqslant n_0$, where

$$P_j = \sum_{i=j}^{\infty} p_i, \quad j \in \mathbb{N}. \tag{2}$$

Proof. One can observe that if $\{z_n\}_{n=1}^{\infty}$ is an eventually negative solution of (E1), then $\{y_n\}_{n=1}^{\infty} = \{-z_n\}_{n=1}^{\infty}$ is the solution of (E1), which is eventually positive. So we prove the theorem for the case y is eventually positive. By virtue of (1) there exists n_1 such that $P_n > 0$ for all $n \geqslant n_1$. Moreover, in this case there exists $n_2 \in \mathbb{N}$ such that $y_n > 0$ for $n \geqslant n_2$. Let us denote $n_0 = \max(n_1, n_2)$. Summing (E1) from $n = n_0$ to $m - 1$ we get

$$\Delta y_m - \Delta y_{n_0} = \sum_{j=n_0}^{m-1} p_j y_j.$$

By using summation by parts and definition (2) we obtain

$$\Delta y_m - \Delta y_{n_0} = -y_m P_m + y_{n_0} P_{n_0} + \sum_{j=n_0}^{m-1} P_{j+1} \Delta y_j.$$

Hence,

$$\Delta y_m + y_m P_m = y_{n_0} P_{n_0} + \sum_{j=n_0}^{m-1} P_{j+1} \Delta y_j + \Delta y_{n_0}, \quad m \geqslant n_0. \tag{3}$$

For $m = n_0 + 1$ we have from (3)

$$\Delta y_{n_0+1} + y_{n_0+1} P_{n_0+1} = \Delta y_{n_0} + y_{n_0} P_{n_0} + P_{n_0+1}(\Delta y_{n_0} + y_{n_0} P_{n_0}) - y_{n_0} P_{n_0} P_{n_0+1}.$$

Hence,

$$\Delta y_{n_0+1} + y_{n_0+1} P_{n_0+1} \leqslant (\Delta y_{n_0} + y_{n_0} P_{n_0})(1 + P_{n_0+1}),$$

and since $y_{n_0+1} P_{n_0+1} > 0$,

$$\Delta y_{n_0+1} \leqslant (\Delta y_{n_0} + y_{n_0} P_{n_0})(1 + P_{n_0+1}).$$

By induction one can show that

$$\Delta y_m + y_m P_m \leqslant (\Delta y_{n_0} + y_{n_0} P_{n_0}) \prod_{j=n_0+1}^{m} (1 + P_j). \tag{4}$$

Hence it follows that (i) holds, where

$$C = \frac{\Delta y_{n_0} + y_{n_0} P_{n_0}}{1 + P_{n_0}}. \tag{5}$$

If y is nondecreasing, then $\Delta y_n \geqslant 0$. So from (4) we get

$$y_n P_n \leqslant C \prod_{j=n_0}^{m} (1 + P_j).$$

Hence (i_1) holds. In a similar way one can prove the cases (ii) and (ii_1). □

Corrollary 1.2. *In the case* (i) *we obtain*

$$y_n \leqslant y_{n_0} + C \sum_{i=n_0}^{n-1} \prod_{j=n_0}^{i} (1 + P_j) = v_n, \quad \textit{for } n \geqslant n_0,$$

and $y_n \geqslant v_n$ *for* $n \geqslant n_0$ *in the case* (ii).

Remark 1.3. Since $P_n > 0$ for $n \geqslant n_0$, by the theorem of convergence of the infinite product $\prod_{j=n_0}^{\infty}(1 + P_j)$ we obtain the following. If the series $\sum_{j=n_0}^{\infty} P_j$ is convergent, then from (i) we get

$$\Delta y_n \leqslant C \prod_{j=n_0}^{\infty} (1 + P_j) = C_1, \quad \text{for } n \geqslant n_0.$$

Hence $y_n \leqslant C_1 n + C_2$ for $n \geqslant n_0$.

Remark 1.4. By assumption (1) there exists $n_3 \in \mathbb{N}$ such that $P_n \in (0, 1)$ for all $n \geqslant n_3$. Denoting in the proof $n_0 = \max(n_1, n_2, n_3)$, we obtain other bounds for solution in the cases (i) and (ii) without the monotonicity assumption of these solutions. From (4) it follows in the case (i) that

$$y_{m+1} - (1 - P_m) y_m \leqslant C \prod_{j=n_0}^{m} (1 + P_j),$$

where C is defined by (5).

Hence by dividing the above inequality by $\prod_{j=n_0}^{m}(1-P_j)$ and substituting

$$z_m = y_m \prod_{j=n_0}^{m-1} (1-P_j)^{-1}, \tag{6}$$

we obtain

$$\Delta z_m \leqslant C \prod_{j=n_0}^{m} \frac{1+P_j}{1-P_j}.$$

Solving this inequality and using (6), we obtain the following bounds of y:

$$y_n \leqslant \left[C_1 + C \sum_{j=n_0}^{n-1} \prod_{i=n_0}^{j} \frac{1+P_i}{1-P_i} \right] \prod_{j=n_0}^{n-1} (1-P_j) = v_n,$$

where C_1 is some constant and C is defined by (5).

In case (ii) we have $y_n \geqslant v_n$.

2. Transformation to the first-order perturbed equation

The difference equations of higher order can be treated in some sense as the equations of first order with perturbed argument. Consider the following equation:

$$\Delta x_n = a_n x_{n+k}, \quad k > 0, \ n \in \mathbb{N}. \tag{E2}$$

Similarly as in [2], with some modifications one can prove the following theorem.

Theorem 2.1. *Let a be a sequence of nonnegative real numbers such that the series* $\sum_{j=1}^{\infty} a_j$ *is convergent. Then there exists a solution x of* (E2) *such that*

$$\lim_{n \to \infty} x_n = C,$$

for any positive constant C.

The asymptotic behavior $\lim_{n\to\infty} z_n = C$ shall be denoted by $z_n = C + \text{o}(1)$.

In the next theorem it will be shown how from asymptotic properties of solutions of difference equation (E2) we are able to get any information on asymptotic behavior of the solution of (E1).

Theorem 2.2. *Let* $p_n < 1$ *for every* $n \in \mathbb{N}$ *and let the series* $\sum_{j=1}^{\infty}(1-p_j)$ *converge. Then for any constant C there exists a solution y of* (E1) *which possesses the asymptotic form*

$$y_n = [C + \text{o}(1)] 2^{-n} \prod_{j=1}^{n-1} (1-p_j).$$

Proof. Equation (E1) can be rewritten in an equivalent form:

$$y_{n+2} - 2y_{n+1} + (1-p_n) y_n = 0.$$

Introducing a new variable

$$z_n = 2^n y_n \prod_{j=1}^{n-1} (1-p_j)^{-1}, \quad n \in \mathbb{N}, \tag{7}$$

equation (E1) can be transformed to the following form:

$$\Delta z_n = \tfrac{1}{4}(1-p_{n+1}) z_{n+2}. \tag{8}$$

This is the equation of type (E2) for $k=2$, where $a_n = \frac{1}{4}(1-p_{n+1})$.

One can observe that the assumptions of Theorem 2.1 are fulfilled. So, (8) possesses a solution with the asymptotic behavior

$$z_n = C + \mathrm{o}(1), \quad \text{where } C \text{ is any positive constant.}$$

Using (7) we obtain the thesis of Theorem 2.2 for positive constant C. Now, applying a similar reasoning as in the proof of Theorem 1.1 we have got this thesis for $C<0$. □

3. Comparison method

The other method to get asymptotic properties of solutions of difference equations consists in comparing two equations of the same order. We shall present such a theorem based on the lemma proved in [5] for the continuous case, and transposed to the difference equations in [6]. The obtained result is of negative character, that means, it selects from any set of sequences some subset to which the solutions of the considered equation do not belong.

Two equations

$$\Delta^2 x_n + p_n x_n = 0, \tag{E3}$$

$$\Delta^2 y_n + b_n y_n^{2t-1} = 0, \tag{E4}$$

where t is some fixed positive integer, will be considered.

[6, Lemma 1] for (E3) and (E4) has the following form.

Lemma 3.1. *Let $p^w = b_n(w_n)^{2t-1}/w_n$, where w belongs to some sequence class. If P is a propositional function defined on a sequence class Z and*

$$S = \{z \in Z\colon z \text{ is a solution of (E4)}\}, \qquad S_p = \{z \in Z\colon z \text{ is a solution of (E3)}\},$$

then

$$(\forall p \in A)(\forall x \in S_p) P(x) \quad \text{and} \quad (\forall y \in S) \sim P(y) \ \Rightarrow\ p^y \in A$$

imply

$$(\forall y \in S) P(y),$$

where A is some sequence class.

Remark 3.2. By [4, Theorem 3] we can say that if $p_n \geqslant 0$ for all $n \in \mathbb{N}$ and the series $\sum_{j=1}^{\infty} j^2 p_j$ is convergent, then every solution of (E3) has the property

$$x_n = C_1 n + C_2 + \mathrm{o}(1), \quad \text{as } n \to \infty,$$

where C_1 and C_2 are constants.

Note added in proof. The authors have been informed by the reviewer that recently Renato Spigler and Marco Vianello at the Università di Padova have obtained error bounds for these asymptotic formulas, even without the assumption $p_n \geqslant 0$.

Now we are able to present the following theorem.

Theorem 3.3. *Let b be a nonnegative sequence such that for some $\tau > 2$, $\tau \in \mathbb{N}$, the series*

$$\sum_{j=1}^{\infty} j^{2t\tau-2\tau+2} b_j \text{ is convergent.} \tag{9}$$

Then for every solution of (E4) *of the form*

$$y_n = \sum_{i=0}^{\tau} C_i n^i + \mathrm{o}(1), \quad n \in \mathbb{N},$$

which never vanishes on $\mathbb{N}$, we have $C_i = 0$ for $i = 2, 3, \ldots, \tau$.

Proof. Let Z denote a class of sequences, which never vanish on $\mathbb{N}$, of the form

$$z_n = \sum_{i=0}^{\tau} C_i n^i + \mathrm{o}(1), \quad n \in \mathbb{N},$$

and A a class of nonnegative sequences such that the series $\sum_{j=1}^{\infty} a_j j^2$ converges for every $a \in A$. By Remark 3.2 we can state that every solution of (E3) is of the form $C_1 n + C_0 + \mathrm{o}(1)$ for an arbitrary sequence $p \in A$. Let the propositional function be

$$p(z)\colon \ z \text{ is of the form } C_1 n + C_0 + \mathrm{o}(1).$$

Suppose that there exists a solution $y \in S$ of (E4) for which the propositional function takes the value false, i.e.,

$$y = \sum_{i=0}^{\sigma} C_i n^i + \mathrm{o}(1), \quad \text{where } \tau \geqslant \sigma > 2 \text{ and } C_\sigma \neq 0.$$

We shall show that p^y belongs to the class A. Let $p^y = \{d_n\}_{n=1}^{\infty} = \{b_n (y_n)^{2t-2}\}_{n=1}^{\infty}$. It is evident that p^y is a nonnegative sequence. Furthermore,

$$\begin{aligned}\sum^{\infty} j^2 d_j &= \sum^{\infty} j^2 \left(\sum_{i=0}^{\sigma} C_i j^i + \mathrm{o}(1) \right)^{2t-2} b_j \\ &\leqslant \sum^{\infty} j^2 b_j \left[\left(\max_{0 \leqslant i \leqslant \sigma} |C_i| \right) \sum_{i=0}^{\sigma} j^i + C \right]^{2t-2} \leqslant \sum^{\infty} \hat{C} b_j j^{2t\sigma - 2\sigma + 2}\end{aligned} \tag{10}$$

for sufficiently large j, where C, $\hat{C}$ are some constants. By the comparison test of convergence from (9), it follows that the series $\sum_{j=1}^{\infty} j^{2t\sigma-2\sigma+2} b_j$ converges and so, by (10), $p^y \in A$. Therefore by Lemma 3.1 every solution of (E4) which is of the type Z is of the form

$$y_n = C_1 n + C_0 + \mathrm{o}(1),$$

and the theorem is proved. $\square$

Remark 3.4. Similar results can be obtained for more general types of equations than (E4). We have formulated the theorem in such a manner for two reasons, namely:
- to present a new and very important type of difference equation (E4);
- to show one of the possible methods which allow to get asymptotic properties of solutions of more complicated equations from the asymptotic behavior of solutions of any simpler one.

4. Asymptotically first-order polynomial solution

Now the asymptotic behavior of solutions of the equation

$$\Delta^2 x_n + p_n f(x_n) = 0 \tag{E5}$$

will be investigated.

A necessary and sufficient condition for some solution x of (E5) to have the asymptotic behavior $x_n = C + o(1)$, $n \in \mathbb{N}$, where C is a constant such that $f(C) \neq 0$ was given in [1].

Here a sufficient condition for some solution x of (E5) to have the asymptotic behavior

$$x_n = n + v_n, \quad n \in \mathbb{N}, \tag{11}$$

where $\{v_n\}$ is a sequence such that $\lim_{n\to\infty} v_n = 0$, will be proved.

It will be assumed that $f : \mathbb{R} \to \mathbb{R}$ is a continuous, one-periodic function such that $f(1) > 0$. Moreover $p : \mathbb{N} \to \mathbb{R}_+$.

Theorem 4.1. *A sufficient condition for the existence of a solution x of* (E5) *which possesses the asymptotic behavior* (11) *is convergence of the series* $\sum_{j=1}^{\infty} jp_j$.

Proof. Let X denote the algebraic sum of the spaces of the sequences c_0 which converge to zero, and the sequences of the form $\{Cn\}_{n=1}^{\infty}$, where $C \in \mathbb{R}$. The set X endowed with the operations addition and multiplication by a scalar (defined as usual for sequences) forms a linear space.

Let $x = \{\xi_n\}_{n=1}^{\infty} \in X$. Let us define the norm

$$\| x \| = \left| \lim_{n\to\infty} \frac{\xi_n}{n} \right| + \sup_{n \geqslant 1} \left| \xi_n - \left(\lim_{n\to\infty} \frac{\xi_n}{n} \right) n \right|. \tag{12}$$

One can verify that X with the norm (12) is a Banach space. Let the series $\sum_{j=1}^{\infty} jp_j$ be convergent. Then, by assumption,

$$\lim_{n\to\infty} \sum_{j=n}^{\infty} jp_j = 0. \tag{13}$$

Moreover, the sequence $\{\sum_{j=n}^{\infty} jp_j\}_{n=1}^{\infty}$ is nonincreasing.

The continuity of f implies that there exists an interval $I = [1-\epsilon, 1]$ such that $f(t) > 0$ for $t \in I$ and for some $\epsilon \in (0, 1)$.

Let $C_1 = \max f(t)$. From (13) it follows that there exists n_1 such that $C_1 \sum_{j=n}^{\infty} jp_j \leqslant \epsilon$ for all $n \geqslant n_1$.

Let $n_2 = \min\{n \in \mathbb{N}\colon C_1 \sum_{j=n}^{\infty} jp_j \leqslant \epsilon\}$. Moreover let us define the set $T \subset X$ in the following way:

$$x = \{\xi_i\}_{i=1}^{\infty} \in T, \quad \text{if } \begin{cases} \xi_k = k, & \text{for } k = 1, 2, \ldots, n_2 - 1, \\ \xi_k \in I_k, & \text{for } k \geqslant n_2, \end{cases}$$

where I_k is the closed interval $[k - C_1 \sum_{j=k}^{\infty} jp_j,\ k]$. The set T is bounded, convex and closed in X.

Using the Hausdorff theorem one can show that T is compact. Let us define on T the operator A as follows:

$$Ax = y = \{\eta_n\}_{n=1}^{\infty}, \quad \text{where } \begin{cases} \eta_k = k, & \text{for } k = 1, 2, \ldots, n_2 - 1, \\ \eta_k = k - \sum\limits_{j=n}^{\infty} (j+1-k) p_j f(\xi_j), & \text{for } k \geqslant n_2. \end{cases}$$

The operator A is a function from T to T. By the assumptions of f we can conclude that f is uniformly continuous on each of the intervals $[n - \epsilon,\ n + \epsilon]$ for $n \in \mathbb{N}$. Moreover, for every $\epsilon > 0$ there exists $\delta > 0$ such that for each $t_1,\ t_2 \in I$ the condition $|t_1 - t_2| < \delta$ implies $|f(t_1 + k) - f(t_2 + k)| < \epsilon$ for all $k \in \mathbb{N}$. Using this fact one can show that A is continuous on T. By the Schauder fixed-point theorem there exists a solution $x = \{\xi_j\}_{j=1}^{\infty}$ in T of the equation $Ax = x$. This equality means that for $n \geqslant n_2$,

$$\xi_n = n - \sum_{j=n}^{\infty} (j+1-n) p_j f(\xi_j). \tag{14}$$

Applying twice the operator Δ to (14) we obtain

$$\Delta^2 \xi_n = -p_n f(\xi_n), \quad \text{for } n \geqslant n_2.$$

Hence the sequence $\{\xi_n\}_{n=1}^{\infty}$ fulfilled (E5) for $n \geqslant n_2$. Let us denote $u_n = \xi_n$ for $n \geqslant n_2$. One can observe that (E5) can be rewritten in the equivalent form

$$x_n + p_n f(x_n) = -x_{n+2} + 2x_{n+1}, \quad n \in \mathbb{N}. \tag{15}$$

Replacing n, x_n, x_{n+1}, x_{n+2} respectively by $n_2 - 1$, x, u_{n_2}, u_{n_2+1}, we get

$$x + p_{n_2-1} f(x) = -u_{n_2+1} + 2u_{n_2}. \tag{16}$$

Since f is a continuous and periodic function on $\mathbb{R}$, the function on the left-hand side of (16) is a surjection on $\mathbb{R}$. Hence there exists a solution of (16). Let us denote this by u_{n_2-1}. Analogously we can calculate $u_{n_2-2}, \ldots, u_2, u_1$. We get the sequence $\{u_n\}_{n=1}^{\infty}$ which fulfills (E5). This sequence is identical with $\{\xi_n\}_{n=1}^{\infty}$ for $n \geqslant n_2$. So it possesses the asymptotic behavior (11). □

Remark 4.2. Let us observe that the function f fulfills the conditions of [3, Theorem 2]. By this theorem (E5) possesses a solution which asymptotically approaches 1. Hence (E5) has two linearly independent solutions z and w such that $z_n = n + o(1)$ and $w_n = 1 + o(1)$.

Remark 4.3. Making some modifications of the set T and operator A, one can show that (E5) under the same set of hypotheses possesses a solution of the form

$$x_n = kn + v_n, \quad \text{where } \lim_{n \to \infty} v_n = 0, \quad k \in \mathbb{N}.$$

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